LU decomposition, where L is a lower-triangular matrix with 1 as the diagonal elements and U is an upper-triangular matrix. Just as there are many combinations of $12 = 1 \cdot 12 = 2 \cdot 6 = 3 \cdot 4 = 4 \cdot 3 = ...$, there are infinite number of combinations of $L \cdot U$. However, when the diagonal elements of L are fixed to be 1, the remaining elements are uniquely fixed.

A linear algebraic equation $A \cdot x = b \rightarrow L \cdot U \cdot x = b \rightarrow L \cdot y = b$ where $U \cdot x = y$

matrix inverse $A^{-1} = (L \cdot U)^{-1} = U^{-1} \cdot L^{-1}$

After LU decomposition, we obtain solution $x$ in a two-step process

Step 0. $A = L \cdot U$
Step 1. Solve $L \cdot y = b \rightarrow y = L^{-1} \cdot b$
Step 2. Solve $U \cdot x = y \rightarrow x = U^{-1} \cdot y$

Example

\[
A := \begin{bmatrix} 0 & 1 & 2 \\ 4 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix}, \quad b := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
\]

$A = L \cdot U$

\[
\begin{bmatrix} 0 & 1 & 2 \\ 4 & 1 & 0 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} \cdot 1 & 0 \\ L_{31} \cdot L_{32} \cdot 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}
\]

work on the 1st row of A

\[
A_{11} = 0 \rightarrow U_{11} = A_{11} = 0
\]
\[
A_{12} = 1 = L_{12} \cdot U_{12} \rightarrow U_{12} = A_{12} = 1
\]
\[
A_{13} = 2 = L_{13} \cdot U_{13} \rightarrow U_{13} = A_{13} = 2
\]

\[
\begin{bmatrix} 1 & 0 & 0 \\ L_{21} \cdot 1 & 0 \\ L_{31} \cdot L_{32} \cdot 1 \end{bmatrix} \begin{bmatrix} U_{11} = 0 & U_{12} = 1 & U_{13} = 2 \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}
\]

work on the 2nd row of A

\[
A_{21} = 4 = L_{21} \cdot U_{11} \rightarrow L_{21} = A_{21} = 4 \rightarrow \frac{U_{11} = 0}{U_{11} = 0} \rightarrow \frac{U_{12} = 1}{U_{11} = 0} \rightarrow \frac{U_{13} = 2}{U_{11} = 0} \quad \text{... divide by 0! ... We stop here!}
\]

\[
\begin{bmatrix} 1 & 0 & 0 \\ L_{21} = 4 & 1 & 0 \\ L_{31} \cdot L_{32} \cdot 1 \end{bmatrix} \begin{bmatrix} U_{11} = 0 & U_{12} = 1 & U_{13} = 2 \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}
\]

For each row, there is a step where we divide by the diagonal element of A. If any of the diagonal element of A is 0, LU decomposition does not exist. Since which equation comes first makes no difference in the solution of $x$, we swap equations, which is equivalent to swapping rows of both A and b.
Pivot. Examine column #1 of all the rows in A, the row with the largest element in this 1st column (in the absolute value sense) becomes the 1st row of the permutated matrix $A'$. Likewise swapping for b.

Examine column #2 of all the rows from row#2 to the last row in A, the row with the largest element in this 2nd column (in the absolute value sense) becomes the 2nd row of the permutated matrix $A'$.

And so on...

\[
A := \begin{pmatrix}
0 & 1 & 2 \\
4 & 1 & 0 \\
1 & 2 & 3
\end{pmatrix}
\quad \Rightarrow \quad A' := \begin{pmatrix}
4 & 1 & 0 \\
1 & 2 & 3 \\
0 & 1 & 2
\end{pmatrix}
\quad \Rightarrow \quad b := \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}
\]

If we work systematically from the first row of $A'$, we can solve for unknown elements in L and U matrices sequentially, each time with only one unknown.

**work on the 1st row of $A'$**

\[
A'_{11} = 4 = 1 \cdot U_{11} \quad \Rightarrow \quad U_{11} = A'_{11} = 4
\]

\[
A'_{12} = 1 = 1 \cdot U_{12} \quad \Rightarrow \quad U_{12} = A'_{12} = 1
\]

\[
A'_{13} = 0 = 1 \cdot U_{13} \quad \Rightarrow \quad U_{13} = A'_{13} = 0
\]

\[
A' = L \cdot U
\]

\[
\begin{bmatrix}
4 & 1 & 0 \\
1 & 2 & 3 \\
0 & 1 & 2
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{21} & 1 & 0 \\
L_{21} & 1 & 0 \\
L_{31} & L_{32} & 1
\end{bmatrix}
\begin{bmatrix}
U_{11} & U_{12} & U_{13} \\
0 & U_{22} & U_{23} \\
0 & 0 & U_{33}
\end{bmatrix}
\]

**work on the 2nd row of $A'$**

\[
A'_{21} = 1 = L_{21} \cdot U_{11} \quad \Rightarrow \quad L_{21} = A'_{21} = 1
\]

\[
A'_{22} = 2 = L_{21} \cdot U_{12} + 1 \cdot U_{22} \quad \Rightarrow \quad U_{22} = A'_{22} - L_{21} \cdot U_{12} = 2 - \frac{1}{4} = \frac{7}{4}
\]

\[
A'_{23} = 3 = L_{21} \cdot U_{13} + 1 \cdot U_{23} \quad \Rightarrow \quad U_{23} = A'_{23} - L_{21} \cdot U_{13} = 3 - \frac{1}{4} = \frac{11}{4}
\]

\[
A' = L \cdot U
\]

\[
\begin{bmatrix}
4 & 1 & 0 \\
1 & 2 & 3 \\
0 & 1 & 2
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{21} & 1 & 0 \\
L_{21} = \frac{1}{4} & 1 & 0 \\
L_{31} & L_{32} & 1
\end{bmatrix}
\begin{bmatrix}
U_{11} & U_{12} & U_{13} \\
0 & U_{22} & U_{23} \\
0 & 0 & U_{33}
\end{bmatrix}
\]

**work on the 3rd row of $A'$**

\[
A'_{31} = 0 = L_{31} \cdot U_{11} \quad \Rightarrow \quad L_{31} = A'_{31} = 0
\]
Thus, $L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ 0 & \frac{4}{7} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 4 & 1 & 0 \\ 0 & \frac{7}{4} & 3 \\ 0 & 0 & \frac{2}{7} \end{bmatrix}$

Step 1. Solve $Ly = b' \rightarrow y = L^{-1}b'$

$A'_{32} = L_{31}'U_{12} + L_{32}'U_{22} \rightarrow \begin{bmatrix} L_{32}' & \frac{A'_{32} - L_{31}'U_{12} = 1 - 0\cdot 1 = \frac{4}{7} & \frac{7}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$A'_{33} = L_{31}'U_{13} + L_{32}'U_{23} + U_{33}' \rightarrow U_{33}' = A'_{33} - L_{31}'U_{13} - L_{32}'U_{23} = 2 - 0\cdot 0 - \frac{4}{7} \cdot \frac{3}{7} = \frac{2}{7}$

$\begin{bmatrix} 4 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} L_{21}' = \frac{1}{4} & 1 & 0 \\ L_{31}' = 0 & L_{32}' = \frac{4}{7} & 1 \end{bmatrix}$

Thus, $L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ 0 & \frac{4}{7} & 1 \end{bmatrix} U = \begin{bmatrix} 4 & 1 & 0 \\ 0 & \frac{7}{4} & 3 \\ 0 & 0 & \frac{2}{7} \end{bmatrix}$

Check: $\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{2}{7} \end{bmatrix}$

$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 0 \\ 0 & \frac{7}{4} & 3 \\ 0 & 0 & 2 \end{bmatrix}$

Step 2. Solve $Ux = y \rightarrow x = U^{-1}y$

$\begin{bmatrix} y_1 = b' = 1 \\ y_2 = b' = \frac{1}{4} \\ y_3 = \frac{2}{7} \end{bmatrix}$

Swapping rows of $A$ does not affect the answer $x$, as long as rows of $b$ are also similarly swapped.
Mathcad's \textbf{lu function} returns 3 matrices: $P$, $L$, $U$ such that $P \cdot A = L \cdot U$.

$P$ is a permutation matrix that has "1" occupying some elements $P_{ij}$ that signifies the raw swapping operation from row $j$ to row $i$.

$$PLU := \text{lu}(A) = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 1 & 0 & 25 & 1 & 0 & 0 & 1.75 \\ 1 & 0 & 0 & 0 & 0.571 & 1 & 0 & 0 & 0.286 \\ \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0.286 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0.571 \\ 1 & 0 & 0 & 0 & 1.75 & 4 & 1 & 0 & 0 \\ \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0.286 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0.571 \\ 1 & 0 & 0 & 0 & 1.75 & 4 & 1 & 0 & 0 \end{pmatrix}$$

$P := \text{submatrix}(PLU, 1, 3, 1, 3)$

$L := \text{submatrix}(PLU, 1, 3, 4, 6)$

$U := \text{submatrix}(PLU, 1, 3, 7, 9)$

\textbf{Pre-multiplication by a permutation matrix = row swapping}

The 1st row of $P$ has $P_{12}=1 \rightarrow$ 2nd row in $A$ goes into 1st row in $A'$.

The 2nd row of $P$ has $P_{23}=1 \rightarrow$ 3rd row in $A$ goes into 2nd row in $A'$.

The 3rd row of $P$ has $P_{31}=1 \rightarrow$ 1st row in $A$ goes into 3rd row in $A'$.

Thus, the permutated matrix $A'$ has: row 2 $\rightarrow$ row 3 $\rightarrow$ row 1 of $A$.

\textbf{check}$$A = \begin{pmatrix} 0 & 1 & 2 \\ 4 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix} \quad \rightarrow \quad PA = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix} \quad \leftarrow \quad \text{compare} \quad \rightarrow \quad LU = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$$

$P$ is orthonormal

$P \cdot P^T = P^T \cdot P = I$

$P^{-1} = P^T$

Applying $P^T$ to the permutated matrix $A'$ reverses the original permutation and yields back the original matrix $A$.

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0.143 & -0.571 & 1 \end{pmatrix} \quad \rightarrow \quad \text{compare} \quad \rightarrow \quad U^{-1} = \begin{pmatrix} 0.25 & -0.143 & 1.5 \\ 0 & 0.571 & -6 \\ 0 & 0 & 3.5 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} -0.5 & 1 & 1.5 \\ 1 & 4 & -6 \\ 0.5 & 2 & 3.5 \end{pmatrix} \quad \leftarrow \quad \text{compare} \quad \rightarrow \quad U^{-1} \cdot L^{-1} = \begin{pmatrix} 0.5 & -1 & 1.5 \\ -1 & 4 & -6 \\ 0.5 & -2 & 3.5 \end{pmatrix}$$

\textbf{Post-multiplication by a permutation matrix = column swapping}

In the equation below, $P^{-1}=P^T$ is also a permutation matrix. Post-multiplying of $A^{-1}$ by $P^{-1}=P^T$ has the following effect:

The 1st column of $P^{-1}$ has $(P^{-1})_{12}=1 \rightarrow$ 2nd column in $A^{-1}$ goes into 1st column in $A^{-1}$.

The 2nd column of $P^{-1}$ has $(P^{-1})_{23}=1 \rightarrow$ 3rd column in $A^{-1}$ goes into 2nd column in $A^{-1}$.

The 3rd column of $P^{-1}$ has $(P^{-1})_{31}=1 \rightarrow$ 1st column in $A^{-1}$ goes into 3rd column in $A^{-1}$.

Thus, the permutated matrix $A^{-1}$ has: column 2 $\rightarrow$ column 3 $\rightarrow$ column 1 of $A^{-1}$.

\textbf{Swapping rows of $A$ results in swapping columns of $A^{-1}$ in the same order.}
\[
A^{-1} = \begin{bmatrix}
1.5 & 0.5 & -1 \\
-6 & -1 & 4 \\
3.5 & 0.5 & -2
\end{bmatrix} \quad \text{compare} \quad A^{-1} \cdot P^{-1} = \begin{bmatrix}
0.5 & -1 & 1.5 \\
-1 & 4 & -6 \\
0.5 & -2 & 3.5
\end{bmatrix} \quad \text{compare} \quad A^{-1} = \begin{bmatrix}
0.5 & -1 & 1.5 \\
-1 & 4 & -6 \\
0.5 & -2 & 3.5
\end{bmatrix}
\]

Effect of swapping rows on matrix inverse

\[l = (P \cdot A) \cdot (P \cdot A)^{-1} = (P \cdot A) \cdot (A^{-1} \cdot P^{-1}) = (P \cdot A) \cdot (A^{-1} \cdot P^T)\]

\[A^{-1} = A^{-1} \cdot P^{-1} = A^{-1} \cdot P^T\]

\[A^{-1} = A^{-1} \cdot P\]

**Post-multiplication by a permutation matrix = column swapping**

In the equation above, post-multiplying of \(A^{-1}\) by \(P\) has the following effect:

The 1st column of \(P\) has \((P)_{31} = 1 \rightarrow 3rd\) column in \(A^{-1}\) goes into 1st column in \(A^{-1}\).

The 2nd column of \(P\) has \((P)_{12} = 1 \rightarrow 1st\) column in \(A^{-1}\) goes into 2nd column in \(A^{-1}\).

The 3rd column of \(P\) has \((P)_{23} = 1 \rightarrow 2nd\) column in \(A^{-1}\) goes into 3rd column in \(A^{-1}\).

Thus, the permutated matrix \(A^{-1}\) has: 2nd column \(\rightarrow\) 3rd column \(\rightarrow\) 1st column of \(A^{-1}\).

From \(A^{-1}\) to \(A^{-1}\), **swap columns** of \(A^{-1}\) in a **reverse** order.

**Post-multiplication by a permutation matrix = column swapping**

In the equation below, post-multiplying of \(A\) by \(P\) has the following effect:

The 1st column of \(P\) has \(P_{31} = 1 \rightarrow 3rd\) column in \(A\) goes into 1st column in \(A''\).

The 2nd column of \(P\) has \(P_{12} = 1 \rightarrow 1st\) column in \(A\) goes into 2nd column in \(A''\).

The 3rd column of \(P\) has \(P_{23} = 1 \rightarrow 2nd\) column in \(A\) goes into 3rd column in \(A''\).

Thus, the permutated matrix \(A''\) has: 3rd column \(\rightarrow\) 1st column \(\rightarrow\) 2nd column of \(A\).

\[
A = \begin{bmatrix}
0 & 1 & 2 \\
4 & 1 & 0 \\
1 & 2 & 3
\end{bmatrix} \quad \text{compare} \quad A'' = A \cdot P \quad A'' = \begin{bmatrix}
2 & 0 & 1 \\
0 & 4 & 1 \\
3 & 1 & 2
\end{bmatrix}
\]
**Gaussian Elimination & LU Decomposition.** Let us illustrate with the same matrix $A$ and vector $b$ as before.

$$
A := \begin{pmatrix}
0 & 1 & 2 \\
4 & 1 & 0 \\
1 & 2 & 3 \\
\end{pmatrix}
\quad
b := \begin{pmatrix}
0 \\
1 \\
0 \\
\end{pmatrix}
$$

Step 0. Augment matrix $A$ and vector $b$

$$
Ab := \text{augment}(A, b) = \begin{pmatrix}
0 & 1 & 2 & 0 \\
4 & 1 & 0 & 1 \\
1 & 2 & 3 & 0 \\
\end{pmatrix}
$$

We represent the steps Gaussian elimination takes in manipulating the elements in the augmented matrix $Ab$ by pre-multiplying with a square matrix, which acts as an operator that operates on the second matrix. Pivoting: swap 1st & 2nd eqn, because eqn (1.2) has the largest leading coefficient:

$$
P_1 := \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\quad
A'b' := P_1 \cdot Ab
\quad
A'b' = \begin{pmatrix}
4 & 1 & 0 & 1 \\
0 & 1 & 2 & 0 \\
1 & 2 & 3 & 0 \\
\end{pmatrix}
$$

* (1.2) by $0/4$ & subtract it from (1.1) $\rightarrow$ (2.2)

* (1.2) by $1/4$ & subtract it from (1.3) $\rightarrow$ (2.3)

$$
G_1 := \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
in the diagonal position for the 1st row of $G_1$ means just transcribe the 1st row of $A'b'$ and do nothing.

- "-0/4" means subtract $0/4$ of 1st row of $A'b'$, and "1" means add $1x$ of 2nd row of $A'b'$.

- "-1/4" means subtract $1/4$ of 1st row of $A'b'$, and "1" means add $1x$ of 3rd row of $A'b'$.

$$
A'b' := G_1 \cdot A'b'
\quad
A'b' = \begin{pmatrix}
4 & 1 & 0 & 1 \\
0 & 1 & 2 & 0 \\
0 & 1.75 & 3 & -0.25 \\
\end{pmatrix}
$$

Pivoting: swap 2nd & 3rd eqn:

$$
P_2 := \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\quad
A'b' := P_2 \cdot A'b'
\quad
A'b' = \begin{pmatrix}
4 & 1 & 0 & 1 \\
0 & 1.75 & 3 & -0.25 \\
0 & 1 & 2 & 0 \\
\end{pmatrix}
$$

* (2.3) by $1/(7/4) & subtract it from (2.2) $\rightarrow$ (3.3)

$$
G_2 := \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
in the diagonal position for the 1st row of $G_2$ means just transcribe the 1st row of $A'b'$ and do nothing.

- "-0/4" means subtract $0/4$ of 1st row of $A'b'$, and "1" means add $1x$ of 2nd row of $A'b'$.

- "-1/4" means subtract $1/4$ of 1st row of $A'b'$, and "1" means add $1x$ of 3rd row of $A'b'$.

$$
A'b' := G_2 \cdot A'b'
\quad
A'b' = \begin{pmatrix}
4 & 1 & 0 & 1 \\
0 & 1.75 & 3 & -0.25 \\
0 & 0 & 0.286 & 0.143 \\
\end{pmatrix}
$$
Below is a minor variation of the above steps where we perform all the pivoting first, rather than pivoting as we go in each step. A combination of two sequential swapping steps is equivalent to pre-multiplying the augmented matrix $Ab$ by $P$, which does multiple swappings in one sweep.

$$P := P_2 P_1 \quad P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad A'b' := P \cdot Ab \quad A'b' = \begin{pmatrix} 4 & 1 & 0 & 1 \\ 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix}$$ (1.2)

* (1.2) by $1/4$ & subtract it from (1.3) $\Rightarrow$ (2.2)
* (1.2) by $0/4$ & subtract it from (1.1) $\Rightarrow$ (2.3)

$$G_1 := \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ 0 & -\frac{4}{1} & 1 \end{pmatrix} \quad A'b' := G_1 \cdot A'b' \quad A'b' = \begin{pmatrix} 4 & 1 & 0 & 1 \\ 1.75 & 3 & -0.25 \\ 0 & 1 & 2 & 0 \end{pmatrix}$$ (2.1)

* (2.2) by $1/(7/4)$ & subtract it from (2.3) $\Rightarrow$ (3.3)

$$G_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{1.75} & 1 \end{pmatrix} \quad A'b' := G_2 \cdot A'b' \quad A'b' = \begin{pmatrix} 4 & 1 & 0 & 1 \\ 1.75 & 3 & -0.25 \\ 0 & 0 & 0.286 & 0.143 \end{pmatrix}$$ (3.1)

We combine the two sequential Gaussian elimination steps $G_1$ & $G_2$ into an equivalent one single operation $G$:

$$G := G_2 G_1 \quad G = \begin{pmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0.143 & -0.571 & 1 \end{pmatrix} \quad A'b' := G \cdot P \cdot Ab \quad A'b' = \begin{pmatrix} 4 & 1 & 0 \\ 1.75 & 3 & -0.25 \\ 0 & 0 & 0.286 & 0.143 \end{pmatrix}$$

The following play on math shows that since the "A" matrix in $A'b$ is upper triangular, the inverse of $G$ is lower triangular and this is the $L$ matrix. Thus, the lower triangular matrix $L$ summarizes all the individual forward elimination steps taken during Gaussian elimination leading up to an upper triangular form, and Gaussian elimination is directly related to LU decomposition.

$$A'b' := G \cdot P \cdot Ab \quad A' := \text{submatrix}(A'b', 1, 3, 1, 3) \quad A' = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 1.75 & 3 \\ 0 & 0 & 0.286 \end{pmatrix}$$

$$G^{-1} \cdot A'b' := P \cdot Ab \quad \text{and} \quad U := A'$$

$$L := G^{-1} \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0.571 & 1 \end{pmatrix} \quad \text{Check:} \quad L \cdot A'b' = \begin{pmatrix} 4 & 1 & 0 & 1 \\ 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix} \quad \begin{pmatrix} 4 & 1 & 0 \\ 4 & 1 & 0 \\ 4 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$$

Check:

$$L \cdot A' = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$$

Check: