Problem 5.36 of Bergman. (Short answer starting on p3 of this worksheet.)
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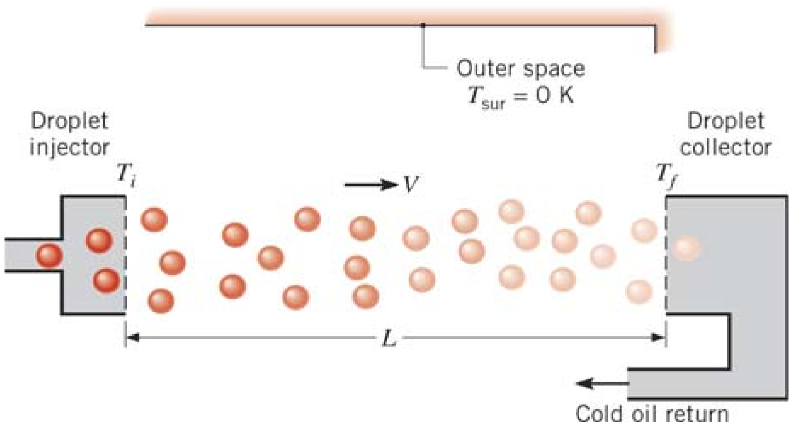
Problem Statement. As permanent space stations increase in size, there is an attendant increase in the amount of electrical power they dissipate. To keep station compartment temperatures from exceeding prescribed limits, it is necessary to transfer the dissipated heat to space. A novel heat rejection scheme that has been proposed for this purpose is termed a Liquid Droplet Radiator (LDR). The heat is first transferred to a high vacuum oil, which is then injected into outer space as a stream of small droplets. The stream is allowed to traverse a distance L, over which it cools by radiating energy to outer space at absolute zero temperature. The droplets are then collected and routed back to the space station.



Consider conditions for which droplets of emissivity ε=0.95 and diameter D=0.5 mm are injected at a temperature of T_i=500K and a velocity of V=0.1m/s. Properties of the oil are ρ=885 kg/m^3, c_P=1900 J/(kg·K), and k=0.145 W/(m·K). Assuming each drop to radiate to deep space at T_{sur}=0K, determine the distance L required for the droplets to impact the collector at a final temperature of T_f=300 K. What is the amount of thermal energy rejected by each droplet?

Thermal properties.

$\rho := 885$ kg/m3 $\quad c_P := 1900$ J/(kg·K) $\quad k := 0.145$ W/(m·K) $\quad \varepsilon := 0.95$ $\quad \sigma := 5.67 \cdot 10^{-8}$ W/(m^2·K^4)

$T_i := 500$ K $\quad T_f := 300$ K $\quad T_{sur} := 0$ K

Physical geometry

$D := 0.5 \cdot 10^{-3}$ m $\quad R := \frac{D}{2}$ $\quad V := 0.1$ m/s

1-dimensional (in r-direction) energy balance equation with heat generation

$$\left(4\cdot\pi\cdot r^2\cdot\Delta r\right)\cdot\rho\cdot c_P\cdot\frac{dT}{dt} = \left(-4\cdot\pi\cdot r^2\cdot k\cdot\frac{dT}{dr}\right)_r - \left(-4\cdot\pi\cdot r^2\cdot k\cdot\frac{dT}{dr}\right)_{r+\Delta r} + \left(4\cdot\pi\cdot r^2\cdot\Delta r\right)\cdot qdot$$

Let Δr go to 0

$$\rho\cdot c_P\cdot\frac{dT}{dt} = \frac{1}{r^2}\cdot\frac{d}{dr}\cdot\left(r^2\cdot k\cdot\frac{dT}{dr}\right) + qdot$$

If k=constant, take the k term out of d()/dr

$$\frac{dT}{dt} = \frac{k}{\rho \cdot c_P} \cdot \frac{1}{r^2} \cdot \frac{d}{dr} \cdot \left(r^2 \cdot \frac{dT}{dr}\right) + \frac{qdot}{\rho \cdot c_P}$$

I.C. $t=0$ $T(r,0) = T_i(r)$

B.C. $r=0$ $\frac{dT(0,t)}{dr} = 0$

$r = R = \frac{D}{2}$ $-\left(4\cdot\pi\cdot R^2\right)\cdot k\cdot \frac{dT(R,t)}{dr} = \left(4\cdot\pi\cdot R^2\right)\cdot\varepsilon\cdot\sigma\cdot\left(T(R,t)^4 - T_{sur}^4\right)$

$$\frac{dT(R,t)}{dr} = -\frac{\varepsilon\cdot\sigma}{k}\cdot\left(T(R,t)^4 - T_{sur}^4\right)$$

qdot=0

$$\frac{dT}{dt} = \frac{k}{\rho \cdot c_P} \cdot \frac{1}{r^2} \cdot \frac{d}{dr} \cdot \left(r^2 \cdot \frac{dT}{dr}\right)$$

non-dimensionalize (dependent variable T)

$$\theta = \frac{T - T_{sur}}{T_i - T_{sur}}$$

θ changes from 1 to 0 with time.

non-dimensionalize (independent variables, r and t)

$$r' = \frac{r}{R} \qquad t' = \frac{t}{\tau_d} \longrightarrow R\cdot dr' = dr \qquad \tau_d\cdot dt' = dt \qquad \left(T_i - T_{surr}\right)\cdot d\theta = dT$$

Substituting dr, dt, and dT into the original dimensional PDE yields,

$$\frac{T_i - T_{sur}}{\tau_d}\cdot\frac{d\theta}{dt'} = \alpha\cdot\frac{1}{R^2\cdot r'^2}\cdot\frac{d}{R\cdot dr'}\cdot\left(R^2\cdot r'^2\cdot\frac{T_i - T_{sur}}{R}\cdot\frac{d\theta}{dr'}\right) \qquad \tau_d = \frac{R^2}{\alpha} = R^2\cdot\frac{\rho\cdot c_P}{k} = \frac{energy\_stored}{heat\_transfer\_rate}$$

$$\frac{d\theta}{dt'} = \frac{1}{r'^2}\cdot\frac{d}{dr'}\cdot\left(r'^2\cdot\frac{d\theta}{dr'}\right)$$

I.C. $t'=0$ $\theta(r',0) = 1$

B.C. $r'=0$ $\frac{d\theta(0,t')}{dr'} = 0$

$r'=1$ $\frac{d\theta(1,t)}{dr'} = -\frac{\varepsilon\cdot\sigma}{k}\cdot\left(T(R,t)^4 - T_{sur}^4\right)$

$$T^4 - T_{sur}^4 = \left(T^2 + T_{sur}^2\right)\cdot\left(T + T_{sur}\right)\cdot\left(T - T_{sur}\right)$$

$$\frac{d\theta(1,t)}{dr'} = -\frac{\varepsilon\cdot\sigma\cdot R}{k}\cdot\left(T^2 + T_{sur}^2\right)\cdot\left(T + T_{sur}\right)\cdot\theta$$

The relevant dimensionless group that is analogous to the Biot number in convection is initially at (which becomes smaller /w time because T decreases /w time):

$$\frac{resistance\_due\_to\_conduction}{resistance\_due\_to\_radiation} = \frac{\varepsilon\cdot\sigma\cdot R}{k}\cdot\left(T_i^2 + T_{sur}^2\right)\cdot\left(T_i + T_{sur}\right) = 0.012$$

(For homework, if we do not check for validity of the lumped capacitance model, we can start right here.) Since the above "Bi-like" number is small, radiation resistance dominates. d θ/dr'=0 at r'=1; constant temperature throughout the droplet; T=T(t) (i.e., T is not a function of r'). We use the following lumped capacitance model.

$$\left(\frac{4}{3}\cdot\pi\cdot R^3\right)\cdot\rho\cdot c_P\cdot\frac{dT}{dt} = -\left(4\cdot\pi\cdot R^2\right)\cdot\varepsilon\cdot\sigma\cdot\left(T^4 - T_{sur}^4\right)$$

length of time it takes to cool from T_i=500K to T_f=300K

$$t := -\frac{R\cdot\rho\cdot c_P}{3\cdot\varepsilon\cdot\sigma}\cdot\int_{T_i}^{T_f}\frac{1}{T^4 - T_{sur}^4}\,dT \qquad t = 25.179 \text{ sec}$$

Short answer -- simply plug numbers into Eqn 5.19 of Bergman (which was derived by following the above heat balance steps)

$$t := \frac{\rho \cdot \left(\frac{4}{3} \cdot \pi \cdot R^3\right) \cdot c_P}{3 \cdot \varepsilon \cdot \left(4 \cdot \pi \cdot R^2\right) \cdot \sigma} \cdot \left(\frac{1}{T_f^{\,3}} - \frac{1}{T_i^{\,3}}\right) \qquad t = 25.179 \quad \text{sec}$$

L=distance the drops traveled $L := V \cdot t \qquad L = 2.518$ m

amount of thermal energy rejected by each droplet

$$\Delta E := \left(\frac{4}{3} \cdot \pi \cdot R^3\right) \cdot \rho \cdot c_P \cdot \left(T_f - T_i\right) \qquad \Delta E = -0.022$$ J/drop ("-" sign means loss of energy)

Comment. What if the Bi-like number is O(1) and we cannot resort to the lumped capacitance model? Note that the B.C. for this problem involves T^4, which is nonlinear; thus, the "exact" solution approach via separation of variables, which is based on eigenvalue-eigenvector, does *not* apply.