Symbolic manipulation. Mathcad offers two means of symbolic calculation, as opposed to numerical calculation: Maple and SmartMath. There are some differences between calling the Maple symbolic engine via the menu and calling SmartMath via the "→" operator. With Maple, as we make changes, we need to go through the menu to obtain updated results. With SmartMath, on the other hand, Mathcad automatically updates the results to reflect the changes, provided that the Automatic Mode is turned on.

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There are 12 items under Mathcad's "Symbolic" menu bar (Version 5).
1. Evaluate (Symbolically, Complex, Floating Point).
2. Simplify.
3. Expand Expression.
4. Factor Expression.
5. Collect on Subexpression.
6. Polynomial Coefficients.
7. Differentiate on Variable.
8. Integrate on Variable.
10. Substitute for Variable.
11. Expand to Series.
12. Convert to Partial Fraction.

It is important to note that if we desire purely symbolic results, make sure that we do not assign any numerical values to variables. The fact that we cannot completely kill a variable once it is created (although we can reset it to zero), often causes conflicts with symbolic operations. Below, we shall briefly explore the capabilities of each of these menu items.

1. Evaluate Symbolically.

As an example, let us consider the following definite integral. Mathcad's integral can handle endpoint singularities, but the precision deteriorates as the problem becomes numerically more challenging.

\[ \int_{0}^{1} x^\alpha \, dx = 2.000000 \quad \leftarrow \text{The answer should be } 1/(\alpha+1)=2 \]

\[ \int_{0}^{1} x^\alpha \, dx = 2.498597 \quad \leftarrow \text{The answer should be } 1/(\alpha+1)=2.5 \]

Eventually, Mathcad completely fails when it becomes a bit more challenging, as shown by the "not converging" error.

\[ \int_{0}^{1} x^\alpha \, dx = \quad \leftarrow \text{The answer should be } 1/(\alpha+1)=1/0.3=3.33... \]

not converging
As a second example, let us consider infinity \( \infty \). Mathcad does pre-define infinity as an extremely large number.

\[
\infty = 1 \cdot 10^{307}
\]

Nevertheless, we need to be careful how we use \( \infty \). Strictly speaking, the symbol \( \infty \) is valid only in a mathematical, symbolic sense. We should carefully guard its use (or more likely its misuse) in a numerical context, let it be limits of integration, an argument to an algebraic operation, or otherwise. The following example should get us to think twice on the numerical use of \( \infty \). What does it mean to have \( \infty \) twice? The difference between infinity is mathematically undefined, but Mathcad gives 0. And infinity plus any finite number should remain infinity. (Although \( 10^{306} \) is indeed very large, it is still finite in a purely mathematical sense.)

\[
\infty + \infty = 2 \cdot 10^{307} \quad \infty - \infty = 0 \quad \infty + 10^{306} = 1.1 \cdot 10^{307} \quad \infty + 10^{306} = 1.1 \cdot \infty
\]

In general, Mathcad does poorly numerically with indefinite integrals although infinity has been numerically pre-defined. (Mathcad almost invariably does miserably with \( \infty \) whenever it is used numerically.)

\[
\int_0^\infty e^{-x} \, dx = \text{not converging}
\]

One option to circumvent the convergence problem is to substitute \( \infty \) with an arbitrary large number (which is already what Mathcad does), but sometimes it not clear as to just how large is large enough to give a numerically meaningful answer without running into convergence problems.

\[
\int_0^{10^4} e^{-x} \, dx = \int_0^{10^3} e^{-x} \, dx = 1
\]

\[
\text{not converging}
\]

In the example above, an upper integration limit of \( 10^4 \) failed but \( 10^3 \) was o.k. Even with convergence, how does one gain confidence that the answer is indeed a correct one? This is when SmartMath comes to the rescue.

\[
\int_0^1 x^\alpha \, dx \rightarrow \text{limit} \left[ \frac{x^{(\alpha+1)}}{\alpha+1} + \frac{1}{\alpha+1}, x=0, \text{right} \right]
\]

\[
\int_0^\infty e^{-x} \, dx \rightarrow 1
\]

Symbolic manipulation can also help. Marking the entire expression and choosing |Symbolic|Evaluate|Evaluate Symbolically| yields respectively for the above two cases:

\[
\int_0^1 x^\alpha \, dx \rightarrow \text{limit} \left[ \frac{x^{(\alpha+1)}}{\alpha+1} + \frac{1}{\alpha+1}, x=0, \text{right} \right]
\]

which further evaluates to:

\[
\lim_{x \rightarrow 0^+} \frac{x^{(\alpha+1)}}{(\alpha+1)} + \frac{1}{(\alpha+1)} \rightarrow 3.33333333333333333333
\]

An example of indefinite integral.

\[
\int_0^\infty e^{-x} \, dx \rightarrow \text{limit} \left[ \frac{x^{(\alpha+1)}}{\alpha+1} + \frac{1}{\alpha+1}, x=0, \text{right} \right]
\]

\[
\int_0^\infty e^{-x} \, dx \rightarrow 1
\]
SymbolicEvaluateSymbolically operates on vectors and matrices.

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \rightarrow \text{SymbolicEvaluateSymbolically} \rightarrow \frac{1}{(a \cdot d - b \cdot c)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}
\]

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \text{SymbolicEvaluateSymbolically} \rightarrow a \cdot d - b \cdot c
\]

\[
\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \rightarrow \text{SymbolicEvaluateSymbolically} \rightarrow \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}
\]

\[
\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \rightarrow \text{SymbolicEvaluateSymbolically} \rightarrow \begin{pmatrix} a_{11} \cdot b_{11} + a_{12} \cdot b_{21} & a_{11} \cdot b_{12} + a_{12} \cdot b_{22} \\ a_{21} \cdot b_{11} + a_{22} \cdot b_{21} & a_{21} \cdot b_{12} + a_{22} \cdot b_{22} \end{pmatrix}
\]

\[
\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1} \rightarrow \text{Symbolic} \rightarrow \frac{1}{(a_{11} \cdot a_{22} \cdot a_{33} - a_{11} \cdot a_{23} \cdot a_{32} - a_{21} \cdot a_{12} \cdot a_{33} + a_{21} \cdot a_{13} \cdot a_{32} + a_{31} \cdot a_{12} \cdot a_{33} - a_{31} \cdot a_{13} \cdot a_{22})} \begin{pmatrix} a_{11} \cdot a_{22} \cdot a_{33} - a_{11} \cdot a_{23} \cdot a_{32} - a_{21} \cdot a_{12} \cdot a_{33} + a_{21} \cdot a_{13} \cdot a_{32} + a_{31} \cdot a_{12} \cdot a_{33} - a_{31} \cdot a_{13} \cdot a_{22} \end{pmatrix}
\]

Similarly, SmartMath also operates on vectors and matrices.

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \rightarrow \frac{1}{(a \cdot d - b \cdot c)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}
\]

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow a \cdot d - b \cdot c
\]

\[
\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \rightarrow \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}
\]

\[
\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \rightarrow \begin{pmatrix} a_{11} \cdot b_{11} + a_{12} \cdot b_{21} & a_{11} \cdot b_{12} + a_{12} \cdot b_{22} \\ a_{21} \cdot b_{11} + a_{22} \cdot b_{21} & a_{21} \cdot b_{12} + a_{22} \cdot b_{22} \end{pmatrix}
\]

\[
\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1} \rightarrow \text{Symbolic} \rightarrow \frac{1}{(a_{11} \cdot a_{22} \cdot a_{33} - a_{11} \cdot a_{23} \cdot a_{32} - a_{21} \cdot a_{12} \cdot a_{33} + a_{21} \cdot a_{13} \cdot a_{32} + a_{31} \cdot a_{12} \cdot a_{33} - a_{31} \cdot a_{13} \cdot a_{22})} \begin{pmatrix} a_{11} \cdot a_{22} \cdot a_{33} - a_{11} \cdot a_{23} \cdot a_{32} - a_{21} \cdot a_{12} \cdot a_{33} + a_{21} \cdot a_{13} \cdot a_{32} + a_{31} \cdot a_{12} \cdot a_{33} - a_{31} \cdot a_{13} \cdot a_{22} \end{pmatrix}
\]
2. Simplify.

Marking the entire expression and choosing |Symbolic|Simplify| yields,

\[
\sin(x)^2 + \cos(x)^2 \rightarrow |\text{Symbolic}|\text{Simplify} | \rightarrow 1
\]

\[
\cosh(x)^2 - \sinh(x)^2 \rightarrow |\text{Symbolic}|\text{Simplify} | \rightarrow 1
\]

\[
\sec(x)^2 - \tan(x)^2 \rightarrow |\text{Symbolic}|\text{Simplify} | \rightarrow 1
\]

Of course, we can try various other trigonometric identities. On the other hand, SmartMath failed to simplify further.

\[
\sin(x)^2 + \cos(x)^2 \rightarrow \sin(x)^2 + \cos(x)^2
\]

3 & 4. Expand and Factor.

|Symbolic|Expand Expression| is good for expressing in powers.

Marking just "(x+2)^2" and choosing |Symbolic|Expand Expression| yields,

\[
(x + 2)^2 + (x + 3)^3 \rightarrow |\text{Symbolic}|\text{Expand Expression} | \rightarrow x^2 + 4x + 9 + (x + 3)^3
\]

Marking the whole expression and choosing |Symbolic|Expand Expression| yields,

\[
(x + 2)^2 + (x + 3)^3 \rightarrow |\text{Symbolic}|\text{Expand Expression} | \rightarrow 10x^2 + 31x + 31 + x^3
\]

Here is another example.

\[
(x - 1)(x + 2)(x + 3) \rightarrow |\text{Symbolic}|\text{Expand Expression} | \rightarrow x^5 + 8x^4 + 21x^3 + 14x^2 - 20x - 24
\]

|Symbolic|Factor Expression| reverses expansion. Although the factored terms may appear in an order different from what we have started, the results are fundamentally identical.

\[
(x + 3)(x - 1)(x + 2)^3 \]

Another interesting, but often overlooked, use of |Symbolic|Factor Expression| is to factor a given number into prime numbers. Prime numbers are important in many mathematical and encryption applications.

123456789 → |Symbolic|Factor Expression| → (3)^5(3803)(3607)

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  a_{11} & a_{22} & a_{33} - a_{11}a_{23}a_{32} - a_{21}a_{12}a_{33} + a_{21}a_{13}a_{32} + a_{31}a_{12}a_{23} - a_{31}a_{13}a_{22}
\end{bmatrix}
\]

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{bmatrix}^T
\rightarrow
\begin{bmatrix}
  a_{11} & a_{21} & a_{31} \\
  a_{12} & a_{22} & a_{32} \\
  a_{13} & a_{23} & a_{33}
\end{bmatrix}
\]
Along the line of prime numbers, |Symbolic|Factor Expression| can turn a decimal number into fractions, provided that there are enough significant digits. All rational numbers have repeating digits.

\[ \frac{8}{7} = 1.142857142857143 \quad \text{The repeating digits are "142857".} \]

\[ 1.142857142857143 \rightarrow |\text{Symbolic}|\text{Factor Expression}| \rightarrow \frac{1142857142857143}{1000000000000000} \quad \text{Insufficient number of significant digits.} \]

\[ 1.1428571428571428571 \rightarrow |\text{Symbolic}|\text{Factor Expression}| \rightarrow \frac{8}{7} \]

Of course, some decimal numbers are exact with finite number of digits (which really means that the repeating trailing digits are "0").

\[ 1.1125 \rightarrow |\text{Symbolic}|\text{Factor Expression}| \rightarrow \frac{89}{80} \]

Some irrational numbers, although lacking repeating digits, can still be factored within the numerical accuracy. Below we try to approximate \( \pi \) with a fraction to within 15, 12, and 4 digits of accuracy.

\[ \pi = 3.141592653589793 \]

\[ 3.141592653589793 \rightarrow |\text{Symbolic}|\text{Factor Expression}| \rightarrow \frac{3254528439}{1035948577} = 3.141592653589793 \]

\[ 3.141592653580 \rightarrow |\text{Symbolic}|\text{Factor Expression}| \rightarrow \frac{3927}{1250} = 3.1416 \]

In fact, the prime number example gives us a hint of what goes on behind |Symbolic|Factor Expression|.

Start with the expression:

\[ \left( \frac{8}{21} \times \frac{7}{6} \right)^2 \]

|Symbolic|Factor Expression| yields: \[ \frac{1}{1764} \left( 16 \times \frac{49}{x} - 49 \right)^2 \quad \text{The smallest whole numbers.} \]

An example of where this factoring technique may be useful is in determining reaction stoichiometry. Marking the whole fraction and choosing |Symbolic|Factor Expression| or |Symbolic|Simplify| yields:

\[ \frac{572}{66} \rightarrow |\text{Symbolic}|\text{Factor Expression}| \rightarrow \frac{26}{3} \]

Or, equivalently, with SmartMath: \[ \frac{572}{66} \rightarrow \frac{26}{3} \]

Thus, the common denominator between 572 and 66 is 572/26=66/3=22.

Marking "572" and choosing |Symbolic|Factor Expression| and repeating the process for "66" yields:

\[ \frac{572}{66} \rightarrow |\text{Symbolic}|\text{Factor}| \rightarrow \frac{(2)^2 \times (11) \times (13)}{66} \rightarrow |\text{Symbolic}|\text{Factor}| \rightarrow \frac{(2)^2 \times (11) \times (13)}{(2) \times (3) \times (11)} \]
5. Collect Terms of Similar Power.

Given the expression: \[ a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 + a^3 + 9a^2b + 27ab^2 + 27b^3 \]

Marking "a" and choosing |Symbolic|Collect on Subexpression| yields,
\[ a^5 + 5a^4b + (10b^2 + 1)a^3 + (10b^3 + 9b)a^2 + (5b^4 + 27b^2)a + b^5 + 27b^3 \]

On the other hand, marking "b" and choosing |Symbolic|Collect on Subexpression| yields,
\[ b^5 + 5a^4b + (27 + 10a^2)b^3 + (27a + 10a^3)b^2 + (5a^4 + 9a^2)b + a^5 + a^3 \]

6. Polynomial Coefficients.

Marking "\sin(x)" and choosing |Symbolic|Polynomial Coefficients| yields
\[
\begin{align*}
a + 2b\sin(x) + 3c\sin^2(x) + 4d\sin^3(x) & \rightarrow |Symbolic|Polynomial Coefficients| \\
\end{align*}
\]

Marking "x" and choosing |Symbolic|Polynomial Coefficients| yields
\[
\begin{align*}
1 + 2x + 3x^2 + 4x^3 & \rightarrow |Symbolic|Polynomial Coefficients| \\
\end{align*}
\]

which, in turn, may be fed into the "polyroots" function for further processing.

\[
\text{polyroots}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} \approx -0.606 \\ \approx -0.072 - 0.638i \\ \approx -0.072 + 0.638i \end{bmatrix}
\]

7 & 8. Differentiation and Integration.

Symbolic Differentiation. Marking "x" and choosing |Symbolic|Differentiate on Variable| yields,
\[
\cos(x)\cdot \sin(x) \rightarrow |Symbolic|Differentiate on Variable| \rightarrow -\sin(x)^2 + \cos(x)^2
\]

Symbolic Integration. Marking "x" and choose |Symbolic|Integrate on Variable| reverses the process.
\[
\cos(x)\cdot \sin(x)
\]

Another way is to mark the entire expression and choose |Symbolic|Evaluate|Evaluate Symbolically|.
\[
\frac{d}{dx} \cos(x)\cdot \sin(x) \rightarrow |Symbolic|Evaluate|Evaluate Symbolically| \rightarrow -\sin(x)^2 + \cos(x)^2
\]
\[
\int (-\sin(x)^2 + \cos(x)^2) \, dx \rightarrow |Symbolic|Evaluate|Evaluate Symbolically| \rightarrow \cos(x)\cdot \sin(x)
\]
Differentiation and integration with SmartMath.

\[
\frac{d}{dx} \cos(x) \cdot \sin(x) \rightarrow \sin(x)^2 + \cos(x)^2 \quad \text{... first order derivative}
\]

\[
\frac{d^2}{dx^2} \cos(x) \cdot \sin(x) \rightarrow -4 \cdot \cos(x) \cdot \sin(x) \quad \text{... second order or higher-order derivatives}
\]

\[
\int (-\sin(x)^2 + \cos(x)^2) \, dx \rightarrow \cos(x) \cdot \sin(x) \quad \text{... integral}
\]

With SmartMath, if we change the definition for \( f(x) \), the derivative expression is automatically updated. (Try changing \( f(x) \) or \( g(x) \) below.) Of course, this is a very desirable attribute.

\[
f(x) := \cos(x) \cdot \sin(x) \quad df_{dx}(x) := \frac{d}{dx} f(x) \quad df_{dx}(x) \rightarrow -\sin(x)^2 + \cos(x)^2
\]

\[
f_{xx}(x) := \frac{d^2}{dx^2} f(x) \quad f_{xx}(x) \rightarrow -4 \cdot \cos(x) \cdot \sin(x)
\]

\[
g(x) := \cos(x)^2 - \sin(x)^2 \quad G(x) := \int_{0}^{x} g(\xi) \, d\xi \quad G(x) \rightarrow \cos(x) \cdot \sin(x)
\]

An example of differentiation with SmartMath is in determining the Jacobian for a set of equations.

\[
f_0(x, y, z) := \sin(x) + y^2 + \ln(z) - 7
\]

\[
f_1(x, y, z) := 3 \cdot x + 2y + 1 - z^3
\]

\[
f_2(x, y, z) := x + y + z - 5
\]

\[
df_{dx}(x, y, z) := \begin{bmatrix}
\frac{d}{dx} f_0(x, y, z) \\
\frac{d}{dx} f_1(x, y, z) \\
\frac{d}{dx} f_2(x, y, z)
\end{bmatrix}
\]

\[
df_{dy}(x, y, z) := \begin{bmatrix}
\frac{d}{dy} f_0(x, y, z) \\
\frac{d}{dy} f_1(x, y, z) \\
\frac{d}{dy} f_2(x, y, z)
\end{bmatrix}
\]

\[
df_{dz}(x, y, z) := \begin{bmatrix}
\frac{d}{dz} f_0(x, y, z) \\
\frac{d}{dz} f_1(x, y, z) \\
\frac{d}{dz} f_2(x, y, z)
\end{bmatrix}
\]

\[
df_{dx}(x, y, z) \rightarrow \begin{bmatrix}
\cos(x) \\
3 \cdot 2^y \cdot \ln(2) \\
1
\end{bmatrix} \quad \text{... Jacobian}
\]

...

Marking "x" and choosing |Symbolic|Solve for Variable| yields the following lengthy analytical expression, which gives all three possible roots.

\[ 1 + 2 \cdot x + 3 \cdot x^2 + 4 \cdot x^3 = 0 \rightarrow |\text{Symbolic}|\text{Solve for Variable}| \rightarrow \]

\[
\left[ -\left( \frac{5}{64} + \frac{5}{144} \cdot \sqrt[3]{6} \right)^{\frac{1}{3}} + \frac{5}{48 \cdot \left( \frac{5}{64} + \frac{5}{144} \cdot \sqrt[3]{6} \right)} - \frac{1}{4} \right]
\]

\[
\left[ -\left( \frac{5}{64} + \frac{5}{144} \cdot \sqrt[3]{6} \right) \cdot \left( \frac{1}{3} \right) + \frac{5}{48 \cdot \left( \frac{5}{64} + \frac{5}{144} \cdot \sqrt[3]{6} \right)} - \frac{1}{4} \right] \cdot \left( \frac{1}{2} \right)
\]

Analytical Solution with SmartMath, which also yields three possible roots.

Given \[ 1 + 2 \cdot x + 3 \cdot x^2 + 4 \cdot x^3 = 0 \] first root \[ \rightarrow | \rightarrow \text{second root} \]

\[ \text{Find}(x) \rightarrow \left[ -\left( \frac{5}{64} + \frac{5}{144} \cdot \sqrt[3]{6} \right)^{\frac{1}{3}} + \frac{5}{48 \cdot \left( \frac{5}{64} + \frac{5}{144} \cdot \sqrt[3]{6} \right)} - \frac{1}{4} \right] \]

10. Substitution.

Substitution. First Mark the following expression and choose |Edit|Copy| \[
\frac{a + b^2}{c + d}
\]

Then, marking "x" and choosing |Symbolic|Substitute for Variable| yields,

\[ 1 + 2 \cdot x + 3 \cdot x^2 + 4 \cdot x^3 \rightarrow |\text{Symbolic}|\text{Substitute for Variable}| \rightarrow 1 + 2 \cdot \frac{(a + b^2)}{(c + d)} + 3 \cdot \frac{(a + b^2)^2}{(c + d)^2} + 4 \cdot \frac{(a + b^2)^3}{(c + d)^3} \]
11. Expand to Series.

Choose |Symbolic|Expand to Series| to obtain Taylor's series. (Mathcad will ask for the order.)

\[
\sin(x) \rightarrow |\text{Symbolic}|\text{Expand to Series}| \rightarrow x - \frac{1}{6} x^3 + \frac{1}{120} x^5 - \frac{1}{5040} x^7 + \frac{1}{362880} x^9 + O(x^{10})
\]

\[
\sinh(x) \rightarrow |\text{Symbolic}|\text{Expand to Series}| \rightarrow x + \frac{1}{6} x^3 + \frac{1}{120} x^5 + \frac{1}{5040} x^7 + \frac{1}{362880} x^9 + O(x^{10})
\]

Although higher versions of Mathcad allow the point of expansion to be specified, Mathcad Version 5 allows only expansion around \(x=0\) with |Symbolic|Expand to Series|. Nevertheless, with just a couple of extra steps, we can expand around any given point \(x_0\). The following example demonstrates expansion of \(\sin(x)\) around \(x_0\), which is not necessarily 0.

Marking "z" and choosing |Symbolic|Expand to Series| yields an expansion around \(z=x-x_0=0\)

\[
\sin(z) \rightarrow |\text{Symbolic}|\text{Expand to Series}| \rightarrow \sin(x_0) + \cos(x_0) \cdot z + \left(\frac{-1}{2} \sin(x_0)\right) \cdot z^2 + \left(\frac{-1}{24} \cos(x_0)\right) \cdot z^3 + \left(\frac{1}{120} \sin(x_0)\right) \cdot z^4 + \left(\frac{-1}{720} \sin(x_0)\right) \cdot z^5 + \cdots
\]

Transform back to \(x\). First, we mark "x-x_0" and copy it into memory via |Edit|Copy|. Subsequently, marking "z" in the above Taylor's series expansion and choosing |Symbolic|Substitute for Variable| yields the Taylor's series expansion around \(x=7\),

\[
z = x - x_0
\]

\[
\sin(x_0) + \cos(x_0) \cdot (x-x_0) - \frac{1}{2} \sin(x_0) \cdot (x-x_0)^2 - \frac{1}{6} \cos(x_0) \cdot (x-x_0)^3 + \frac{1}{24} \sin(x_0) \cdot (x-x_0)^4 + \frac{1}{120} \cos(x_0) \cdot (x-x_0)^5 + \cdots
\]

Check: (We first define the Taylor's series approximation by copying the above expression.

\[
\sin_{\text{Taylor}}(x,0) := \sin(x_0) + \cos(x_0) \cdot (x-x_0) - \frac{1}{2} \sin(x_0) \cdot (x-x_0)^2 - \frac{1}{6} \cos(x_0) \cdot (x-x_0)^3 + \frac{1}{24} \sin(x_0) \cdot (x-x_0)^4 + \frac{1}{120} \cos(x_0) \cdot (x-x_0)^5 + \cdots
\]

The Taylor's series approximation is accurate when \(x\) is near the point of expansion.

\[
x := 2 \quad \sin_{\text{Taylor}}(x,0) = 0.909 \quad \sin(x) = 0.909
\]

It fails when \(x\) is far away from the point of expansion.

\[
x := 10 \quad \sin_{\text{Taylor}}(x,0) = 1.448 \cdot 10^5 \quad \sin(x) = -0.544
\]

To reach a better approximation, we move the point of expansion closer to \(x\).

\[
\sin_{\text{Taylor}}(x,8) = -0.544 \quad \sin(x) = -0.544
\]

12. Partial Fraction. Partial fraction is valuable in integrating complicated expressions. It is also useful in finding the Laplace or Fourier transforms for complicated expressions.

\[
\frac{1 + 3 \cdot x}{(2 + 5 \cdot x)^2} + \frac{2}{3 + 2 \cdot x} + \frac{3 + x}{(1 + x) \cdot (1 - x)}
\]

Marking "x" and choosing |Symbolic|Convert to Partial Fraction| yields,

\[
\frac{-1}{5 \cdot (2 + 5 \cdot x)^2} + \frac{3}{5 \cdot (2 + 5 \cdot x)} + \frac{2}{3 + 2 \cdot x} + \frac{1}{1 + x} - \frac{2}{x - 1}
\]