Eigenvector-eigenvector of the second derivative operator $d^2/dx^2$. This leads to Fourier series (sine, cosine, Legendre, Bessel, Chebyshev, etc). This is an example of a systematic way of generating a set of mutually orthogonal basis vectors via the eigenvalues-eigenvectors to an operator. We are often less interested in the solution of the ODE or the second derivative operator itself. Main objective: we are often more interested in the resulting set of eigenvectors that possess “nice” properties (orthogonality, recursive relationships, derivative & integral relationships, etc).

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Problem Statement. Find the eigenvalues and eigenvectors (eigenfunctions) for the second derivative operator $L$ defined in $x=[-1,1]$. We will use the terms eigenvectors and eigenfunctions interchangeably because functions are a type of vectors.

$$L \cdot y = D^2 \cdot y = \frac{d^2}{d x^2} \cdot y = \lambda \cdot y \quad y(1) = y(-1) = 1 \quad \text{or any symmetric boundary condition}$$

By “symmetric BC” above, we mean it as a part of a symmetric linear transformation $L$ that does not change scalar product upon swapping an input vector to the operator. Do not confuse the word “symmetric” with an “even” function symmetrical wrt to $x=0$)

$$\langle L \cdot y, z \rangle = \langle y, L \cdot z \rangle \quad \ldots \text{definition of a symmetric linear transformation (we swap any } y \text{ and } z \text{ in the LVS)}$$

The function that satisfies the differential portion is:

$$y = \exp(\sqrt{\lambda} \cdot x) \quad \text{Check: } \frac{d^2}{d x^2} \exp(\sqrt{\lambda} \cdot x) = \lambda \cdot y$$

If we loosely consider just the ODE, there are an infinity number of eigenvalues and infinite number of eigenvectors -- basically any exponential function $\exp(\lambda'x)$ with any constant $\lambda'$ will do. Specifically, the derivative operator comes with initial conditions or boundary conditions. If we apply the given symmetrical boundary condition and confine ourselves strictly to one-dimensional invariant subspace, there is no eigenvector that satisfies the boundary condition. If we relax the one-dimensional invariant subspace restriction and allow two-dimensional invariant subspace to be considered, we have the following eigenvalues and eigenvectors (depending on the sign of $\lambda'$).

For $\lambda' > 0$:

$$y = \exp(\sqrt{\lambda'} \cdot x) \quad \text{and} \quad y = \exp(-\sqrt{\lambda'} \cdot x)$$

which is equivalent to a combination of

$$y = \sinh(\sqrt{\lambda'} \cdot x) \quad \text{and} \quad y = \cosh(\sqrt{\lambda'} \cdot x)$$

For $\lambda' = 0$:

$$y = 1 \quad \text{and} \quad y = x$$

For $\lambda' < 0$:

$$y = \exp(\sqrt{-\lambda'} \cdot x) = \exp(-\sqrt{-\lambda'} \cdot x) = \exp(\sqrt{-1} \cdot \lambda' \cdot x) \quad y = \exp(-\sqrt{-\lambda'} \cdot x) = \exp(-\sqrt{-1} \cdot \lambda' \cdot x)$$

$$y = \exp(i \cdot \sqrt{-1} \cdot \lambda' \cdot x) \quad \text{and} \quad y = \exp(-i \cdot \sqrt{-1} \cdot \lambda' \cdot x) \quad \text{where} \quad -\lambda' = \sqrt{\lambda'}$$

The square root of a negative number is an imaginary number. The exponential functions with imaginary arguments, $\exp(i \cdot \sqrt{-1} \cdot \lambda' \cdot x)$ and $\exp(-i \cdot \sqrt{-1} \cdot \lambda' \cdot x)$, are equivalent to a combination of

For $\lambda' < 0$:

$$y = \sin(\lambda' \cdot x) \quad \& \quad y = \cos(\lambda' \cdot x) \quad \text{because} \quad e^{i \cdot \sqrt{-1} \cdot \lambda' \cdot x} = \cos(\lambda' \cdot x) + i \cdot \sin(\lambda' \cdot x)$$
In general, for each eigenvalue of the second derivative operator, there are two linearly independent, but not necessarily orthogonal eigenvectors. The two eigenvectors is the result of being in a two-dimensional invariant subspace. (Remember, two-dimensional space means, by definition, that there are two linearly independent vectors.) For each eigenvalue, the eigenvectors are not unique. We can have different combinations of two eigenvectors, and the resulting linearly independent vectors are also eigenvectors. Of these eigenvectors, hyperbolic sine, and x are odd functions (antisymmetric with respect to x=0, f(-x)=-f(x)) that do not satisfy the symmetric boundary condition of the differential equation. Hyperbolic cosine function is even (symmetric with respect to x=0, f(-x)=f(x)), but it has a minimum value of 1 at x=0 and does not satisfy the B.C. Only the cosine and sine functions are valid. Of these, only the cosine functions satisfy y(-1)=y(1)=1, and only the sine functions satisfy y(-1)=y(1)=0. Without loss of generality, we can write the eigenvalue-eigenvector equation in terms of $-\lambda^2$, rather than simply $\lambda'$. Doing so saves us from having to write the square root sign. Thus, strictly speaking, the set $\{\lambda'\}$ are the eigenvalues for the second-derivative operator. However, we commonly refer to the set $\{\lambda\}$ as the eigenvalues; it is just a more general use of the term. To satisfy the symmetric B.C., we must choose $\lambda$ to be integer multiples of $\pi$.

For $\lambda_0=0$, $y_0=1$

For $\lambda_j=\pi j$ where $j=1,2,\ldots,\infty$, $y_j=\cos(\lambda_j x)$

After eliminating sinh and cosh functions, there still remain an infinite number of eigenvalues and an equally infinite number of the corresponding eigenvectors. Although we first discuss cosine eigenvectors, sine eigenvectors are similar. Cosine eigenvectors are all mutually orthogonal to one another; furthermore, they are already normalized (except for $\cos(0^*x)=1$). The following are a few examples.

Define scalar product $\prod(f, g) := \int_{-1}^{1} f(x) \cdot g(x) \, dx$

\[
\int_{-1}^{1} \cos(1\cdot\pi\cdot x) \cdot \cos(2\cdot\pi\cdot x) \, dx = 0
\]

\[
\int_{-1}^{1} \cos(1\cdot\pi\cdot x) \cdot \cos(3\cdot\pi\cdot x) \, dx = 0
\]

\[
\int_{-1}^{1} \cos(2\cdot\pi\cdot x) \cdot \cos(3\cdot\pi\cdot x) \, dx = 0
\]

In general,

For $j=0$

\[
\int_{-1}^{1} 1 \, dx = 2 \quad \rightarrow \text{Normalize} \rightarrow \quad y_0 = \frac{1}{\sqrt{2}}
\]

For $j=k$

\[
\int_{-1}^{1} \cos(j\cdot\pi\cdot x) \cdot \cos(k\cdot\pi\cdot x) \, dx = 0
\]

\[
\int_{-1}^{1} y_j(x) \cdot y_k(x) \, dx = 0
\]

For $j=1,2,\ldots,\infty$

\[
\int_{-1}^{1} \cos(j\cdot\pi\cdot x) \cdot \cos(j\cdot\pi\cdot x) \, dx = 1
\]

\[
\int_{-1}^{1} y_j(x) \cdot y_j(x) \, dx = 1
\]
Now, with these eigenvectors, we can express any twice-differentiable even function in $x=[-1,1]$ as a linear combination of these eigenvectors. This is a huge, nontrivial statement, because it means the series has to converge for the statement to hold, and we can state it with confidence even without any kind of convergence test or formal proof. (Actually, it turns out, with rigorous proof, the sort that mathematicians like, that we can relax the LVS to include all continuous functions that has at most a finite number of discontinuities in $x=[-1,1]$.) Since there are an infinite number of eigenvectors, we need an infinity number of terms in theory (pure mathematics). However, we have to truncate to a finite number of terms $N$ in practice (engineering mathematics).

$$f = \sum_{j=0}^{\infty} a_j \cdot \frac{1}{\sqrt{2}} + \sum_{j=1}^{\infty} a_j \cdot \cos(\lambda_j \cdot x) \quad \text{in practice} \quad f = a_0 \cdot \frac{1}{\sqrt{2}} + \sum_{j=1}^{N} a_j \cdot \cos(\lambda_j \cdot x)$$

Since the eigenvectors are all mutually orthogonal, the following projection formula apply.

$$a_j = \frac{\int_{-1}^{1} f(x) \cdot \cos(\lambda_j \cdot x) \, dx}{\int_{-1}^{1} \cos^2(\lambda_j \cdot x) \, dx}$$

The cosine eigenvectors are also normalized; thus, we can eliminate the denominator.

$$a_j = \frac{\int_{-1}^{1} f(x) \cdot \cos(\lambda_j \cdot x) \, dx}{\int_{-1}^{1} \cos^2(\lambda_j \cdot x) \, dx}$$

We apply the second derivative operator and estimate the second derivative of any twice-differentiable function in $x=[-1,1]$ that satisfies $f(-1)=f(1)=1$. The second derivative of $f$ is,

$$\frac{d^2}{dx^2} f = \sum_{j=0}^{\infty} a_j \cdot \frac{d^2}{dx^2} y_j = \sum_{j=0}^{\infty} a_j \cdot \frac{d^2}{dx^2} y_j = \sum_{j=0}^{\infty} -a_j \cdot (\lambda_j)^2 \cdot y_j = \sum_{j=1}^{\infty} -a_j \cdot (\lambda_j)^2 \cdot y_j$$
Example. Express a constant function \( f(x) = 1 \) as a linear combination of the eigenvectors.

\[
1 = a_0 \frac{1}{\sqrt{2}} + \sum_{j=1}^{\infty} a_j \cos (\lambda_j x) \quad \text{where} \quad a_j = \left< f, y_j \right> = \int_{-1}^{1} f(x) \cos (\lambda_j x) \, dx
\]

Let us approximate \( f(x) = 1 \) with limited number of terms.

\[
N := 10 \quad j := 1 \ldots N \quad f(x) := 1
\]

\[
\lambda_0 := 0 \quad a_0 := \int_{-1}^{1} f(x) \frac{1}{\sqrt{2}} \, dx \quad a_0 = 1.414 \quad \ldots \text{"average" of } f(x)
\]

\[
\lambda_j := \pi j \quad a_j := \int_{-1}^{1} f(x) \cos (\lambda_j x) \, dx
\]

\[
f_{\text{approx}}(x) := a_0 \frac{1}{\sqrt{2}} + \sum_{j=1}^{N} a_j \cos (\lambda_j x)
\]

Actually, since \( f(x) = 1 = y_0 \), which is in turn orthogonal to all the cosine eigenvectors, all the coefficients other than \( a_0 \) are 0. Thus, this case is rather trivial. The point of this exercise is to demonstrate how we can represent any twice-differentiable function as a linear combination of the eigenvectors. It is always a good practice to start with something simple to make sure the method works. We try another function below.
Example. Express the following function as a linear combination of the eigenvectors.

\[
\alpha := 25 \quad f(x) := \frac{1}{1 + \alpha x^2} \quad \rightarrow \text{rewrite as} \rightarrow \quad f(x) = a_0 \frac{1}{\sqrt{2}} + \sum_{j=1}^{\infty} a_j \cos(\lambda_j x)
\]

\[
N := 5 \quad j := 1 \ldots N \quad \leftarrow \text{Change the number of terms } N \text{ and see how the approximation changes.}
\]

\[
\lambda_0 := 0 \quad a_0 := \int_{-1}^{1} f(x) \frac{1}{\sqrt{2}} \, dx \quad a_0 = 0.388
\]

\[
\lambda_j := \pi j \quad a_j := \int_{-1}^{1} f(x) \cos(\lambda_j x) \, dx
\]

\[
f_{\text{approx}}(x) := a_0 \frac{1}{\sqrt{2}} + \sum_{j=1}^{N} a_j \cos(\lambda_j x)
\]

The first derivative of the given function is,

Analytical solution: \( f'(x) := \frac{-2 \cdot \alpha \cdot x}{(1 + \alpha \cdot x^2)^2} \)

Linear combination of eigenvectors: \( f'_{\text{approx}}(x) := \sum_{j=1}^{N} -a_j (\lambda_j) \sin(\lambda_j x) \)

The second derivative of the given function is,

Analytical solution: \( f''(x) := 2 \cdot \alpha \cdot \frac{3 \cdot \alpha \cdot x^2 - 1}{(1 + \alpha \cdot x^2)^3} \)

Linear combination of eigenvectors: \( f''_{\text{approx}}(x) := \sum_{j=1}^{N} -a_j (\lambda_j)^2 \cos(\lambda_j x) \)
**Example.** Express the following Bessel's function as a linear combination of the eigenvectors.

\[
f(z) := J_0(z) \quad \text{for } z = [\alpha, \beta] \quad \alpha := 0 \quad \beta := 20 \quad \rightarrow \text{rewrite as} \quad \rightarrow \quad f(x) = a_0 \frac{1}{\sqrt{2}} + \sum_{j=1}^{\infty} a_j \cos(\lambda_j x)
\]

\[xx(z) := \frac{2(z - \alpha)}{\beta - \alpha} - 1 \quad z(x) := \frac{\beta - \alpha}{2}(x + 1) + \alpha\]

\[N := 10 \quad j := 1 \ldots N \quad \leftarrow \text{Change the number of terms } N \text{ and see how the approximation changes.}\]

\[\lambda_0 := 0 \quad a_0 := \int_{-1}^{1} f(z(x)) \cdot \frac{1}{\sqrt{2}} \, dx\]

\[\lambda_j := \pi j \quad a_j := \int_{-1}^{1} f(z(x)) \cdot \cos(\lambda_j x) \, dx \quad b_j := \int_{-1}^{1} f(z(x)) \cdot \sin(\lambda_j x) \, dx\]

\[f_{\text{approx}}(x) := a_0 \frac{1}{\sqrt{2}} + \sum_{j=1}^{N} a_j \cos(\lambda_j x) + \sum_{j=1}^{N} b_j \sin(\lambda_j x) \quad \alpha := \alpha + 0.1 \ldots \beta\]
Actually the eigenvalues $\lambda$ do not need to be integer multiples of $\pi$. The eigenvectors only need to satisfy the following general symmetric B.C. in the Sturm-Liouville problem. And there are many ways to satisfy it.

$$p\left(y'_m y_n - y'_n y_m\right)_{\lambda=a} = p\left(y'_m y_n - y'_n y_m\right)_{\lambda=b} \quad p=1 \quad a=-1 \quad b=1$$

One way is: \( y'=c \cdot y \) at \( x=a=-1 \) and \( y'=c \cdot y \) at \( x=b=1 \), where \( c \) is any constant.

Another way is: \( y'=c \cdot y \) at \( x=a=-1 \) and \( y'=d \cdot y \) at \( x=b=1 \).

$$p\left[(c \cdot y'_m y_n - (c \cdot y'_n y_m)\right]_{\lambda=a} = p\left[(d \cdot y'_m y_n - (d \cdot y'_n y_m)\right]_{\lambda=b} \quad \text{... this will involve both sine and cosine.}$$

Example. \( c := 2 \quad \rightarrow \cos(\lambda \cdot x) \) alone (without \( \sin(\lambda \cdot x) \)) or \( \sin(\lambda \cdot x) \) alone will do.

$$\lambda_0 := \text{lambda}(1) \quad \lambda_1 = 1.077 \quad N := 10 \quad j := 0 \ldots N - 1 \quad \lambda_j + 1 := \text{lambda}\left(\lambda_j + \pi\right)$$

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Check orthogonality

$$j := 0 \ldots N \quad k := 0 \ldots N \quad \text{coscos}_{j,k} := \int_{-1}^{1} \cos(\lambda_j \cdot x) \cdot \cos(\lambda_k \cdot x) \, dx$$

Approximate

$$f(x) := 1 \quad \int_{-1}^{1} f(x) \cdot \cos(\lambda_j \cdot x) \, dx$$

$$a_j := \frac{1}{\int_{-1}^{1} \cos(\lambda_j \cdot x) \cdot \cos(\lambda_j \cdot x) \, dx} \quad \int_{-1}^{1} f(x) \cdot \cos(\lambda_j \cdot x) \, dx$$

$$f_{\text{approx}}(x) := \sum_{j=0}^{N} a_j \cdot \cos(\lambda_j \cdot x)$$

Because \( c \) can be any constant, the number of possible eigenvalue-eigenvector sets is infinite.
Fourier Series. In the above development, we applied the idea of eigenvalue-eigenvector to represent any given vector from the LVS. For a general function (not necessarily odd nor even), we keep both the cosine and sine eigenvectors. In addition, for simplicity, we take the eigenvector for $\lambda = 0$ to be 1 (which is not normalized), and shift the normalization factor of 2 to the $a_0$ term. In mathematical literature, this is called the Fourier series. When the infinite series is truncated to a finite number of terms, it is called the truncated Fourier series.

$$f(x) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos(\lambda_j x) + \sum_{j=1}^{\infty} b_j \sin(\lambda_j x)$$

where

- $\lambda_j = \pi j$ for $j = 1, 2, \ldots, \infty$
- $a_0 = \int_{-1}^{1} f(x) \, dx$
- $a_j = \langle f, y_j \rangle = \int_{-1}^{1} f(x) \cdot \cos(\lambda_j x) \, dx$
- $b_j = \langle f, z_j \rangle = \int_{-1}^{1} f(x) \cdot \sin(\lambda_j x) \, dx$

Cosine Series. For even functions, keep only the cosine terms.

$$f(x) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos(\lambda_j x)$$

where

- $\lambda_j = \pi j$ for $j = 1, 2, \ldots, \infty$
- $a_0 = \int_{-1}^{1} f(x) \, dx$
- $a_j = \langle f, y_j \rangle = \int_{-1}^{1} f(x) \cdot \cos(\lambda_j x) \, dx$

Sine Series. For odd functions, keep only the sine terms.

$$f(x) = \frac{a_0}{2} + \sum_{j=1}^{\infty} b_j \sin(\lambda_j x)$$

where

- $\lambda_j = \pi j$ for $j = 1, 2, \ldots, \infty$
- $b_0 = \int_{-1}^{1} f(x) \, dx$
- $b_j = \langle f, z_j \rangle = \int_{-1}^{1} f(x) \cdot \sin(\lambda_j x) \, dx$

Note that $a_0$ is the average of the given function $f(x)$ within the interval $x=[-1, 1]$. For problems where the given interval is not within $x=[-1, 1]$, we can always transform the given interval from $x=[\alpha, \beta]$ to $x=[-1, 1]$. The example above demonstrates this.

Fourier Series in Complex Form. cos and sin can be expressed in terms of exp, and vice versa.

$$e^{i \cdot \theta} = \cos(\theta) + i \cdot \sin(\theta)$$

Euler's formula

$$\cos(\theta) = \frac{e^{i \cdot \theta} + e^{-i \cdot \theta}}{2} \quad \sin(\theta) = \frac{e^{i \cdot \theta} - e^{-i \cdot \theta}}{2i}$$

where $i^2 = -1$ is the imaginary number, not to be confused with an index.

Substituting the above cos and sin expressions into the Fourier series formula, we obtain,

$$f(x) = \frac{a_0}{2} + \sum_{j=1}^{\infty} \frac{\exp(i \cdot \lambda_j x) + \exp(-i \cdot \lambda_j x)}{2} + \sum_{j=1}^{\infty} \frac{\exp(i \cdot \lambda_j x) - \exp(-i \cdot \lambda_j x)}{2i}$$

$$f(x) = \frac{1}{2} \left( A_0 + \sum_{j=1}^{\infty} A_j \exp(i \cdot \lambda_j x) + \sum_{j=1}^{\infty} B_j \exp(-i \cdot \lambda_j x) \right)$$

$$A_0 = a_0 \quad A_j = a_j - i \cdot b_j \quad B_j = a_j + i \cdot b_j$$
\[ A_j = \int_{-1}^{1} f(x) \, dx \quad A_j = \int_{-1}^{1} f(x) \cdot \exp(-i \cdot \lambda_j \cdot x) \, dx \quad B_j = \langle f, z_j \rangle = \int_{-1}^{1} f(x) \cdot \exp(i \cdot \lambda_j \cdot x) \, dx \]

Note that \( A_j \) is the coefficient for the \( \exp(i \cdot \lambda_j \cdot x) \) term, but \( A_j \) is the scalar product of \( f(x) \) and \( \exp(-i \cdot \lambda_j \cdot x) \), not \( \exp(i \cdot \lambda_j \cdot x) \). Likewise for \( B_j \). Why do we interchange the terms in the integral that defines the scalar product? On the surface, it seems this is not following the projection formula. However, we are dealing with complex numbers here, and a scalar product definition that is valid for real Euclidean space is invalid for complex space. We need to modify the definition slightly for complex space so that it obeys the general rules of scalar products.

\[
(f, g) = \int_{\alpha}^{\beta} \overline{f(x)} \cdot g(x) \, dx \quad \text{where } \overline{f(x)} \text{ is the complex conjugate of } f(x).
\]

\[
= \Re(f) - i \cdot \Im(f)
\]

or

\[
(f, g) = \int_{\alpha}^{\beta} f(x) \cdot \overline{g(x)} \, dx
\]

**Detour.** For columns of complex numbers, either one of the following definitions may be suitable (but the definition is not unique).

\[
(x, y)^T = \overline{x} \cdot y \quad \text{or} \quad \overline{x} \cdot y = x^T \overline{y} \quad \text{where } \overline{x} \text{ is the complex conjugate of } x.
\]

**Example.**

\[
x := \begin{pmatrix} 1 \\ i \end{pmatrix}
\]

Definition #1.

\[
(x, y)^T = \overline{x} \cdot y \\
\overline{x} \cdot x = 2 \\
(1 - i) \cdot \begin{pmatrix} 1 \\ i \end{pmatrix} = 2 \quad \text{... ok}
\]

Definition #2.

\[
(x, y)^T = x^T \overline{y} \\
(x)^T \overline{x} = 2 \\
(1 \cdot i) \cdot \begin{pmatrix} 1 \\ -i \end{pmatrix} = 2 \quad \text{... ok}
\]

Definition #3 (bad).

\[
(x, y)^T = (x)^T \cdot y \\
x^T \cdot x = 0 \\
(1 \cdot i) \cdot \begin{pmatrix} 1 \\ i \end{pmatrix} = 0 \quad \text{... bad example.}
\]

The 3rd definition is invalid, because the length of a nonzero vector should be positive for the definition to be valid.

**Example.** The above two definitions give two different scalar products that are complex conjugate of each other. However, the lengths from both definitions are identical.

\[
x := \begin{pmatrix} 1 + i \\ 2 + i \cdot 3 \end{pmatrix} \quad y := \begin{pmatrix} 1 + i \\ -1 - i \cdot 3 \end{pmatrix}
\]

Definition #1.

\[
(x, y)^T = \overline{x} \cdot y \\
\overline{x} \cdot x = -9 - 3i \\
(\overline{x})^T \cdot x = 15 \\
\overline{y} \cdot y = 12
\]

Definition #2.

\[
(x, y)^T = x^T \overline{y} \\
x^T \overline{x} = -9 + 3i \\
x^T \overline{x} = 15 \\
y^T \overline{y} = 12
\]

Complex conjugate ↑

↑ Identical ↑
The eigenfunctions $\exp(i \lambda_j^* x)$ and $\exp(-i \lambda_j^* x)$ are mutually orthogonal.

\[
\langle \exp(i \lambda_j^* x), \exp(i \lambda_k^* x) \rangle = \int_{-1}^{1} \exp(i \lambda_j^* x) \cdot \exp(-i \lambda_k^* x) \, dx = 0
\]

\[
\langle \exp(i \lambda_j^* x), \exp(-i \lambda_j^* x) \rangle = \int_{-1}^{1} \exp(i \lambda_j^* x) \cdot \exp(-i \lambda_j^* x) \, dx = 1
\]

\[
\langle \exp(-i \lambda_j^* x), \exp(-i \lambda_k^* x) \rangle = \int_{-1}^{1} \exp(-i \lambda_j^* x) \cdot \exp(-i \lambda_k^* x) \, dx = 0
\]

\[
\langle \exp(-i \lambda_j^* x), \exp(i \lambda_j^* x) \rangle = \int_{-1}^{1} \exp(-i \lambda_j^* x) \cdot \exp(i \lambda_j^* x) \, dx = 0
\]

We can further compact the linear combination expression into a simpler form.

\[
f(x) = \frac{1}{2} \sum_{j=-\infty}^{\infty} A_j \cdot \exp(i \lambda_j^* x) \quad A_j = \int_{-1}^{1} f(x) \cdot \exp(-i \lambda_j^* x) \, dx
\]

Or, equivalently,

\[
f(x) = \frac{1}{2} \sum_{j=-\infty}^{\infty} B_j \cdot \exp(-i \lambda_j^* x) \quad B_j = \int_{-1}^{1} f(x) \cdot \exp(i \lambda_j^* x) \, dx
\]

In the Fourier series approach, we break up a given function into various frequency components. The coefficient $A_j$ tells us how much of the given function $f(x)$ can be expressed as a sinusoidal oscillation of frequency $\lambda_j$. Thus, Fourier series allows us to extract the various frequency components imbedded in the given function. The set of coefficients $\{A_j\}$ are the amplitudes at the corresponding frequencies, and the whole set is commonly referred to as the spectrum of $f(x)$. Fourier series is very important in many aspects of engineering analysis: harmonic oscillation, spectral analysis, digital signal processing, noise filtering, ...

A closely related idea is Fourier transform, which, too, breaks a given continuous function into continuous frequency components, rather than just at frequencies corresponding to eigenvalues of a harmonic oscillator. We extend the integration limit between $-\infty$ and $+\infty$ and replace the summation sign in the last expression with an integral in a continuous domain.

\[
\mathcal{F}(f(x)) = F(\omega) = \int_{-\infty}^{\infty} \exp(i \cdot 2 \pi \omega \cdot x) \cdot f(x) \, dx \quad \text{forward Fourier transform (analogous to the coefficients $\{A_j\}$.)}
\]

\[
\mathcal{F}^{-1}(F(\omega)) = f(x) = \int_{-\infty}^{\infty} \exp(-i \cdot 2 \pi \omega \cdot x) \cdot F(\omega) \, d\omega \quad \text{Inverse Fourier transform (analogous to the linear combination equation.)}
\]
Many other versions of Fourier transform and inverse transform pairs exist.

\[ \mathcal{F}(f(x)) = \mathcal{F}(\omega) = \int_{-\infty}^{\infty} \exp(-i \cdot 2\pi \cdot \omega \cdot x) \cdot f(x) \, dx \]

\[ \mathcal{F}(\mathcal{F}(\omega))^{-1} = \mathcal{F}(x) = \int_{-\infty}^{\infty} \exp(i \cdot 2\pi \cdot \omega \cdot x) \cdot \mathcal{F}(\omega) \, d\omega \]

In the following pair, we factor out the \(2^*\pi\) term and distribute it equally between the forward and inverse transform pairs.

\[ \mathcal{F}(f(x)) = \mathcal{F}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-i \cdot \omega \cdot x) \cdot f(x) \, dx \]

\[ \mathcal{F}(\mathcal{F}(\omega))^{-1} = \mathcal{F}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(i \cdot \omega \cdot x) \cdot \mathcal{F}(\omega) \, d\omega \]

In the following pair, we shift the \(2^*\pi\) term to the inverse transform. We can also shift the \(2^*\pi\) term to the forward transform.

\[ \mathcal{F}(f(x)) = \mathcal{F}(\omega) = \int_{-\infty}^{\infty} \exp(-i \cdot \omega \cdot x) \cdot f(x) \, dx \]

\[ \mathcal{F}(\mathcal{F}(\omega))^{-1} = \mathcal{F}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(i \cdot \omega \cdot x) \cdot \mathcal{F}(\omega) \, d\omega \]

Our Fourier series here most closely resembles the following transform pairs.

\[ \mathcal{F}(f(x)) = \mathcal{F}(\omega) = \int_{-\infty}^{\infty} \exp(-i \cdot \pi \cdot \omega \cdot x) \cdot f(x) \, dx \]

\[ \mathcal{F}(\mathcal{F}(\omega))^{-1} = \mathcal{F}(x) = \frac{1}{2} \int_{-\infty}^{\infty} \exp(i \cdot \pi \cdot \omega \cdot x) \cdot \mathcal{F}(\omega) \, d\omega \]

Discretized version of Fourier transform:

\[ F_j = \frac{1}{\sqrt{n}} \sum_{k=0}^{n} f_k \cdot \exp\left(i \cdot 2\pi \cdot \frac{j \cdot k}{n}\right) \quad \ldots \text{Forward Fourier transform} \]

\[ f_j = \frac{1}{\sqrt{n}} \sum_{k=0}^{n} F_k \cdot \exp\left(-i \cdot 2\pi \cdot \frac{j \cdot k}{n}\right) \quad \ldots \text{Inverse Fourier transform} \]

Examples of Fourier transforms: Our ears and the associated nerve cells convert mechanical sound waves in the time-domain into frequency (pitch in sound) and the associated amplitude (loudness). Likewise, our eyes transform electromagnetic waves in the visible region into frequency (i.e., colors) and the associated amplitude (brightness). In terms of common household items, a radio extracts a specific frequency component, and the television also works in a similar fashion.
Fourier series is a method of expressing a given function with linearly independent functions taken from the same LVS; it is not a linear transform. In other words, Fourier series, extended to infinite number of terms, is just a representation of the same function; it does not change one function to another. It is expressing a vector with a set of linearly independent basis vectors. Fourier transform, on the other hand, converts one function to another function. In other words, it is a function of functions (vectors); thus, Fourier transform is a linear transform.

Fourier series is analogous to Taylor’s series expansion of a given function. Taylor’s series is based on the various derivatives evaluated at one single point. Whereas, the Fourier series is based on various integrals over an interval. Taylor’s series is exact at one single point around which expansion is made, and the approximation degrades as we move farther away from that point of expansion. Fourier series gives a good approximation over the entire interval.